The computational and decisional Diffie-Hellman assumptions in CryptoVerif

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CryptoVerif is a prover for security protocols that
- is sound in the computational model
- produces proofs by sequences of games
- can give asymptotic or exact security results
- provides a generic method for specifying assumptions on cryptographic primitives

Our goal: extend CryptoVerif to Diffie-Hellman key agreements.
- an important primitive;
- difficult for handle in formal protocol provers.
Decisional Diffie-Hellman assumption

Consider a multiplicative cyclic group $G$ of order $q$, with generator $g$. A probabilistic polynomial-time adversary has a negligible probability of distinguishing

$$(g^a, g^b, g^{ab}) \text{ for random } a, b \in \mathbb{Z}_q^*$$

and

$$(g^a, g^b, g^c) \text{ for random } a, b, c \in \mathbb{Z}_q^*$$
Decisional Diffie-Hellman assumption in CryptoVerif

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$$(g^a, g^b, g^c)$$

for random $a, b, c \in \mathbb{Z}_q^*$

In CryptoVerif,

$$\forall i \leq N \text{ new } a : Z; \text{ new } b : Z;$$

$$\quad (OA() := \exp(g, a), OB() := \exp(g, b), ODH() := \exp(g, \text{ mult}(a, b)))$$

$$\approx$$

$$\forall i \leq N \text{ new } a : Z; \text{ new } b : Z; \text{ new } c : Z;$$

$$\quad (OA() := \exp(g, a), OB() := \exp(g, b), ODH() := \exp(g, c))$$
Decisional Diffie-Hellman assumption in CryptoVerif

\[ \forall i \leq N \quad \text{new } a : Z; \text{new } b : Z; \]
\[ (OA() := \exp(g, a), OB() := \exp(g, b), ODH() := \exp(g, \text{mult}(a, b))) \]
\[ \approx \]

\[ \forall i \leq N \quad \text{new } a : Z; \text{new } b : Z; \text{new } c : Z; \]
\[ (OA() := \exp(g, a), OB() := \exp(g, b), ODH() := \exp(g, c)) \]

We replace \( g^{ab} \) with \( g^c \) for some fresh random number \( c \), provided \( a \) and \( b \) are random numbers used only in \( g^a \), \( g^b \), and \( g^{ab} \).

Computational Diffie-Hellman assumption

Consider a multiplicative cyclic group $G$ of order $q$, with generator $g$. A probabilistic polynomial-time adversary has a negligible probability of computing $g^{ab}$ from $g, g^a, g^b$, for random $a, b \in \mathbb{Z}_q^*$. 
Computational Diffie-Hellman assumption in CryptoVerif

Consider a multiplicative cyclic group $G$ of order $q$, with generator $g$. A probabilistic polynomial-time adversary has a negligible probability of computing $g^{ab}$ from $g$, $g^a$, $g^b$, for random $a, b \in \mathbb{Z}_q^*$. In CryptoVerif, this can be written

\[
\begin{align*}
!i \leq N & \text{ new } a : \mathbb{Z}; \text{ new } b : \mathbb{Z}; \\
(OA()) & := \exp(g, a), \ OB() := \exp(g, b), \\
!i' \leq N' \ OCDH(z : G) & := z = \exp(g, \text{mult}(a, b)))
\end{align*}
\]

\[
\approx
\begin{align*}
!i \leq N & \text{ new } a : \mathbb{Z}; \text{ new } b : \mathbb{Z}; \\
(OA()) & := \exp(g, a), \ OB() := \exp(g, b), \\
!i' \leq N' \ OCDH(z : G) & := \text{false}
\end{align*}
\]
Computational Diffie-Hellman assumption in CryptoVerif

Consider a multiplicative cyclic group $G$ of order $q$, with generator $g$. A probabilistic polynomial-time adversary has a negligible probability of computing $g^{ab}$ from $g$, $g^a$, $g^b$, for random $a, b \in \mathbb{Z}_q^*$. In CryptoVerif, this can be written

$$\forall i \leq N \; \text{new} \; a : \mathbb{Z}; \text{new} \; b : \mathbb{Z};$$

$$(\text{OA}() := \exp(g, a), \text{OB}() := \exp(g, b),$$

$$\forall i' \leq N' \; \text{OCDH}(z : G) := z = \exp(g, \text{mult}(a, b)))$$

$$\approx$$

$$\forall i \leq N \; \text{new} \; a : \mathbb{Z}; \text{new} \; b : \mathbb{Z};$$

$$(\text{OA}() := \exp(g, a), \text{OB}() := \exp(g, b),$$

$$\forall i' \leq N' \; \text{OCDH}(z : G) := \text{false})$$

Application: semantic security of hashed El Gamal in the random oracle model (A. Chaudhuri).
Typical protocol using the Diffie-Hellman key agreement

Assumptions on primitives:
- CDH + h is a hash function in the random oracle model
- or DDH + h is an entropy extractor

A simplified form of a Diffie-Hellman key agreement protocol:

Message 1. A → B: \( g^a \), for random a
Message 2. B → A: \( g^b \), for random b

The shared key is \( h(g^{ab}) = h((g^a)^b) = h((g^b)^a) \)

(Signatures omitted for simplicity.)
Typical protocol using the Diffie-Hellman key agreement in CryptoVerif

\[
\begin{align*}
!^{iA \leq N} & \text{cA}(); \new a : Z; \overline{\text{cA}}\langle \exp(g, a) \rangle; \text{cA}(gb)\text{.let } k = h(\exp(gb, a)) \in \ldots \\
\mid
\end{align*}
\]

\[
\begin{align*}
!^{iB \leq N} & \text{cB}(); \new b : Z; \overline{\text{cB}}\langle \exp(g, b) \rangle; \text{cA}(ga)\text{.let } k = h(\exp(ga, b)) \in \ldots \\
\mid
\end{align*}
\]

\[
!^{iH \leq nH} \text{cH}(x); \overline{\text{cH}}\langle h(x) \rangle
\]

Cannot be transformed by the previous CDH/DDH equivalences, because \(a\) and \(b\) are chosen in parallel processes, not one after the other under the same replication.
Extending the formalization of CDH in CryptoVerif

After applying the security assumption on the hash function $h$,
- $h(x)$ returns a fresh random number if $h(x)$ has not already been called,
- and the same result as the previous call otherwise.

Hence $h(x)$ is replaced with lookups that compare $x$ with the other arguments of $h$.

\[
\begin{align*}
!i_A \leq N \quad & cA(); \textbf{new} \ a : Z; \overline{cA}(\exp(g, a)); \overline{cA}(gb) \ldots \exp(gb[u], a[u]) = \exp(gb, a) \\
& \ldots \exp(ga[u'], b[u']) = \exp(gb, a) \ldots x[u''] = \exp(gb, a) \\
| \\
!i_B \leq N \quad & cB(); \textbf{new} \ b : Z; \overline{cB}(\exp(g, b)); \overline{cA}(ga) \ldots \exp(gb[u], a[u]) = \exp(ga, b) \\
& \ldots \exp(ga[u'], b[u']) = \exp(ga, b) \ldots x[u''] = \exp(ga, b) \\
| \\
!i_H \leq nH \quad & cH(x); \ldots \exp(gb[u], a[u]) = x \ldots \exp(ga[u'], b[u']) = x \ldots x[u''] = x.
\end{align*}
\]
Extending the formalization of CDH in CryptoVerif

\[\new a : Z; (\text{OA}(a) := \exp(g, a), \text{Oa}(a) := a, \new b : Z; (\text{OB}(b) := \exp(g, b), \text{Ob}(b) := b, \]

\[\text{CDH}(m : G, j \leq nb) := m = \exp(g, \text{mult}(b[j], a))), \]

\[\text{CDH}(m : G, j \leq na) := m = \exp(g, \text{mult}(a[j], b))) \approx \]

\[\new a : Z; (\text{OA}(a) := \exp(g, a), \text{Oa}(a) := a, \new b : Z; (\text{OB}(b) := \exp(g, b), \text{Ob}(b) := b, \]

\[\text{CDH}(m : G, j \leq nb) := \]

\[\text{if Ob}[j] \text{ or Oa has been called then}
  \]

\[m = \exp(g, \text{mult}(b[j], a)) \]

\[\text{else false),} \]

\[\new b : Z; (\text{OB}(b) := \exp(g, b), \text{Ob}(b) := b, \]

\[\text{CDH}(m : G, j \leq na) := (\text{symmetric of CDH}(a))) \]
Extending the formalization of CDH in CryptoVerif

\[ !^{ia \leq na} \textbf{new } a : Z; (OA() := \exp(g, a), Oa() := a, \]
\[ !^{iaCDH \leq naCDH} \text{OCDHa}(m : G, j \leq nb) := m = \exp(g, \text{mult}(b[j], a))) \]
\[ !^{ib \leq nb} \textbf{new } b : Z; (OB() := \exp(g, b), Ob() := b, \]
\[ !^{ibCDH \leq nbCDH} \text{OCDHb}(m : G, j \leq na) := m = \exp(g, \text{mult}(a[j], b))) \]
\[ \approx \]
\[ !^{ia \leq na} \textbf{new } a : Z; (OA() := \exp(g, a), Oa() := \text{let } ka = \text{mark in } a, \]
\[ !^{iaCDH \leq naCDH} \text{OCDHa}(m : G, j \leq nb) := \]
\[ \text{find } u \leq nb \text{ such that defined}(kb[u], b[u]) \land b[j] = b[u] \text{ then } \]
\[ m = \exp(g, \text{mult}(b[j], a)) \]
\[ \text{else if defined}(ka) \text{ then } m = \exp(g, \text{mult}(b[j], a)) \text{ else false), } \]
\[ !^{ib \leq nb} \textbf{new } b : Z; (OB() := \exp(g, b), Ob() := \text{let } kb = \text{mark in } b, \]
\[ !^{ibCDH \leq nbCDH} \text{OCDHb}(m : G, j \leq na) := (\text{symmetric of OCDHa}) \]
Extending the formalization of CDH in CryptoVerif

\[ \text{!}^{ia \leq na} \text{new} \ a : Z \ ; \ (OA()) := \exp(g, a), Oa()[3] := a, \]
\[ \text{!}^{iaCDH \leq naCDH} \ \text{OCDHa}(m : G, j \leq nb)[\text{required}] := m = \exp(g, \text{mult}(b[j]), \]
\[ \text{!}^{ib \leq nb} \text{new} \ b : Z \ ; \ (OB()) := \exp(g, b), Ob()[3] := b, \]
\[ \text{!}^{ibCDH \leq nbCDH} \ \text{OCDHb}(m : G, j \leq na) := m = \exp(g, \text{mult}(a[j], b))), \]
\[ \approx (\#OCDHa + \#OCDHb) \times \max(1, e^2 \#Oa) \times \max(1, e^2 \#Ob) \times \]
\[ p\text{CDH}(\text{time} + (na + nb + \#OCDHa + \#OCDHb) \times \text{time(exp))} \]
\[ \text{!}^{ia \leq na} \text{new} \ a : Z \ ; \ (OA()) := \exp'(g, a), Oa() := \text{let} \ ka = \text{mark in} \ a, \]
\[ \text{!}^{iaCDH \leq naCDH} \ \text{OCDHa}(m : G, j \leq nb) := \]
\[ \text{find} \ u \leq nb \ \text{such that } \text{defined}(kb[u], b[u]) \land b[j] = b[u] \ \text{then} \]
\[ m = \exp(g, \text{mult}(b[j], a)) \]
\[ \text{else if} \ \text{defined}(ka) \ \text{then} \ m = \exp'(g, \text{mult}(b[j], a)) \ \text{else} \ \text{false}, \]
\[ \text{!}^{ib \leq nb} \text{new} \ b : Z \ ; \ (OB()) := \exp'(g, b), Ob() := \text{let} \ kb = \text{mark in} \ b, \]
\[ \text{!}^{ibCDH \leq nbCDH} \ \text{OCDHb}(m : G, j \leq na) := (\text{symmetric of OCDHa})) \]
Other declarations for Diffie-Hellman (1)

\( g : G \)  generator of \( G \)
\( \exp(G, Z) : G \)  exponentiation
\( \text{mult}(Z, Z) : Z \)  commutative product in \( \mathbb{Z}^* \)
\( \exp(\exp(z, a), b) = \exp(z, \text{mult}(a, b)) \)  \((z^a)^b = z^{ab}\), equal by commutativity of \( \text{mult} \)

\((g^a)^b = g^{ab}\) and \((g^b)^a = g^{ba}\), equal by commutativity of \( \text{mult} \)

\((\exp(g, x) = \exp(g, y)) = (x = y)\)
\((\exp'(g, x) = \exp'(g, y)) = (x = y)\)

Injectivity

\( \text{new } x_1 : Z; \text{new } x_2 : Z; \text{new } x_3 : Z; \text{new } x_4 : Z; \)
\( \text{mult}(x_1, x_2) = \text{mult}(x_3, x_4) \approx_{1/|Z|} \text{false} \)
\((\text{mult}(x, y) = \text{mult}(x, y')) = (y = y')\)

Collision between products
Other declarations for Diffie-Hellman (2)

$$!i \leq N \text{ new } X : G; OX() := X$$
$$\approx_0 \text{ [manual] } !i \leq N \text{ new } x : Z; OX() := \exp(g, x)$$

This equivalence is very general, apply it only manually.

$$!i \leq N \text{ new } X : G; (OX() := X, !i' \leq N' \text{ OXm}(m : Z)[\text{required}] := \exp(X, m))$$
$$\approx_0$$

$$!i \leq N \text{ new } x : Z; (OX() := \exp(g, x), !i' \leq N' \text{ OXm}(m : Z) := \exp(g, \text{mult}(x, m))$$

This equivalence is a particular case applied only when $X$ is inside $\exp$, and good for automatic proofs.

$$!i \leq N \text{ new } x : Z; OX() := \exp(g, x)$$
$$\approx_0 !i \leq N \text{ new } X : G; OX() := X$$

And the same for $\exp'$. 
The implementation of the support for CDH required two extensions of CryptoVerif:

- An array index $j$ occurs as argument of a function.
  - extend the language of equivalences used for specifying assumptions on primitives.
- The equality test $m = \exp(g, \text{mult}(b, a))$ typically occurs inside the condition of a \texttt{find}.
  - This \texttt{find} comes from the transformation of a hash function in the Random Oracle Model.

After transformation, we obtain a \texttt{find} inside the condition of a \texttt{find}. 
Extending the formalization of DDH in CryptoVerif

\[
\begin{align*}
\text{\texttt{ia}} \leq \text{\texttt{na}} & \quad \text{new } a : \mathbb{Z}; (O_A() := \exp(g, a), O_a() := a, \\
\text{\texttt{iaDH}} \leq \text{\texttt{naDH}} & \quad \text{ODH_a}(j \leq \text{\texttt{nb}}) := \exp(g, \text{\texttt{mult}}(b[j], a))), \\
\text{\texttt{ib}} \leq \text{\texttt{nb}} & \quad \text{new } b : \mathbb{Z}; (O_B() := \exp(g, b), O_b() := b, \\
\text{\texttt{ibDH}} \leq \text{\texttt{nbDH}} & \quad \text{ODH_b}(j \leq \text{\texttt{na}}) := \exp(g, \text{\texttt{mult}}(a[j], b))). \\
\approx & \\
\text{\texttt{ia}} \leq \text{\texttt{na}} & \quad \text{new } a : \mathbb{Z}; (O_A() := \exp(g, a), \\
& \quad O_a() := \text{if ODH_a or ODH_b(ia) has been called and returned \\
& \quad ca or cb then event abort else a,} \\
\text{\texttt{iaDH}} \leq \text{\texttt{naDH}} & \quad \text{ODH_a}(j \leq \text{\texttt{nb}}) := \\
& \quad \text{if Ob[j] or Oa has been called then exp(g, mult(b[j], a)) else \\
& \quad \text{if cb or ca defined for b[j], a then that cb or ca else \\
& \quad \texttt{new ca : G; ca),} \\
\text{\texttt{ib}} \leq \text{\texttt{nb}} & \quad \text{symmetric of a's case)}
\end{align*}
\]
Extending the formalization of DDH in CryptoVerif

\[ \ldots \approx \]

\[ \forall a \leq n_a \textbf{ new } a : Z ; (O_A) := \exp(g, a), \]

\[ O_a() := \]

\[ \text{find } u_a' \leq n_a' \text{ st } \text{def}(k'_a[u'_a]) \text{ then event abort else} \]
\[ \text{find } u_b' \leq n_b', u_b \leq n_b \text{ st } \text{def}(k'_b[u'_b], u_b, a'[u'_b, u_b]) \land a'[u'_b, u_b] = a \text{ then} \]
\[ \text{event abort else} \]
\[ \text{let } k_a = \text{mark in } a, \]

\[ \forall a' \leq n_{a'} \textbf{ ODHa}(j \leq n_b) := \textbf{let } b' = b[j] \textbf{ in} \]
\[ \text{find } u \leq n_b \text{ st } \text{def}(k_b[u], b[u]) \land b' = b[u] \text{ then } \exp(g, \text{mult}(b', a)) \text{ else} \]
\[ \text{if } \text{def}(k_a) \text{ then } \exp(g, \text{mult}(b', a)) \text{ else} \]
\[ \text{let } k'_a = \text{mark in} \]
\[ \text{find } v_a' \leq n_{a'} \text{ st } \text{def}(b'[v_a'], c_a[v_a']) \land b' = b'[v_a'] \text{ then } c_a[v_a'] \text{ else} \]
\[ \text{find } v_b' \leq n_b', v_b \leq n_b \text{ st } \text{def}(b[v_b], a'[v_b', v_b], c_b[v_b', v_b]) \land b' = b[v_b] \land \]
\[ a = a'[v_b', v_b] \text{ then } c_b[v_b', v_b] \text{ else} \]
\[ \textbf{new } c_a : G; c_a, \]

\[ \forall b \leq n_b \text{ (symmetric of } a \text{'s case) } \]
Extensions for DDH

The implementation of the support for DDH required two extensions of CryptoVerif:

- An array index $j$ occurs as argument of a function.
  - Already done for CDH.
- Support for abort events in the right-hand side of equivalences.
  - Informally, when $R$ contains abort events, the assumption $L \approx R$ means that $L$ is indistinguishable from $R$ provided the abort events are not executed!
  - More formally, the assumption $L \approx_p R$ implies $C[L] \approx_{p'} C[R]$ with

$$p'(t) = \max_{C' \text{ in time } t} \Pr \left[ C'[C[R]] \rightsquigarrow \text{abort} \right] + p(t + t_C)$$

When an abort event is executed, the adversary can distinguish $C[R]$ from the $C[L]$.
- CryptoVerif will try to show that abort events have a negligible probability of being executed.
This formalization of Diffie-Hellman is included in the library of primitives of CryptoVerif: one can use it without redefining it.

It has been used for proving protocols:

- Signed Diffie-Hellman key agreement
  - DDH + entropy extractor
  - CDH + random oracle model
- One-encrypted key exchange (OEKE, variant of EKE)
  - CDH + random oracle model + ideal cipher model

It can obviously still prove

- El Gamal (DDH)
- Hashed El Gamal
  - DDH + entropy extractor
  - CDH + random oracle model